$\mathbb{N}$ to $\mathbb{Z}$

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Of course $\exists\mathbb{N}$. That much is easily proven; we have in every possible world-state w the existence of the empty set, $\emptyset \subset w$. From there we take power sets. It is a theorem that the cardinality of a power set of some set A, $|\wp(A)| = 2^{|A|}$, 2 taken to the power of the cardinality of the original set. Now every set is disjoint with its power set; $\forall B \subset A$, we have $B \in \wp(A)$. There is a theorem in number theory that every natural number $n \in \mathbb{N}$ may be expressed as $n = x_0 2^{p_0} + x_1 2^{p_1} + \cdots + x_{j-1} 2^{p_{j-1}} + x_j 2^{p_j}$, where $\{x_0, \ldots, x_j\} \subset \{0, 1\}$ and $\{p_0 \ldots p_j\} \subset \{x \in \mathbb{N} : 2^x \leq n\}$. Then $\forall n \in \mathbb{N}$, $\exists A$ such that $|A| = n$.

Now let us create an ordering on $\mathbb{N}$. We have $\forall n \in \mathbb{N}$ and $\forall m \in \mathbb{N}$, $n \leq m$ if, when $|A| = n$ and $|B| = m$, $A \setminus B = \emptyset$. We can make a slightly stronger statement: if $A \setminus B \neq \emptyset$, then $m < n$. Unfortunately this does not account for the existence of negative numbers, which are required for $\mathbb{Z}$. So now we take an arbitrary $o \in \mathbb{N}$ such that for any given $n \in \mathbb{N}$, $n < o$. In other words, o is arbitrarily greater than $|\emptyset|$, and of course $\forall A$, $|\emptyset| \leq |A|$. We will refer to this value o as "0" or "zero".

But it is a theorem that $\mathbb{N}$ is an unbounded set, so $\forall x \in \mathbb{N}$, $\exists y \in \mathbb{N}$ such that $x < y$. Then there are infinitely many values greater than 0 as well. Now we have a solid ordering on our number line; all that remains to create $\mathbb{Z}$ is to define the negative and positive numbers. A number x is *positive* if $0 < x$; a number x is negative if $x < 0$. Therefore $\exists\mathbb{Z}$.

Now we are only two functions away from completeness under the field axioms and the creation of $\mathbb{Q}$ proper.